The Stability of Walrasian General Equilibium

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Abstract

We prove the stability of equilibrium in a completely decentralized Walrasian general equilibrium economy in which prices are fully controlled by economic agents, with production and trade occurring out of equilibrium.

Journal of Economic Literature Classifications:
C62—Existence and Stability Conditions of Equilibrium
D51—Exchange and Production Economies
D58—Computable and Other Applied General Equilibrium Economies

1 Introduction

Walras (1954 [1874]) developed a general model of competitive market exchange, but provided only an informal argument for the existence of a market-clearing equilibrium for this model. Wald (1951 [1936]) provided a proof of existence for a simplified version of Walras' model, and this proof was substantially generalized by Debreu (1952), Arrow & Debreu (1954), Gale (1955), Nikaido (1956), McKenzie (1959), Negishi (1960), and others.

The stability of the Walrasian economy was a central research focus in the years following the existence proofs (Arrow and Hurwicz 1958, 1959, 1960; Arrow, Block and Hurwicz 1959; Nikaido 1959; McKenzie 1960; Nikaido and Uzawa 1960). Following Walras' tâtonnement process, these models assumed that there is no production or trade until equilibrium prices are attained, and out of equilibrium, there is a price profile shared by all agents, the time rate of change of which is a function of excess demand. These efforts at proving stability were successful only by assuming narrow and implausible conditions (Fisher 1983). Indeed,

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Scarf (1960) provided simple examples of unstable Walrasian equilibria under a tâtonnment dynamic.

Several researchers then explored the possibility that allowing trading out of equilibrium could sharpen stability theorems (Uzawa 1959, 1961, 1962; Negishi 1961; Hahn 1962; Hahn and Negishi 1962, Fisher 1970, 1972, 1973), but these effort enjoyed only limited success. Moreover, Sonnenschein (1973), Mantel (1974, 1976), and Debreu (1974) showed that any continuous function, homogeneous of degree zero in prices, and satisfying Walras' Law, is the excess demand function for some Walrasian economy. These results showed that no general stability theorem could be obtained based on the tâtonnement process. Indeed, subsequent analysis showed that chaos in price movements is the generic case for the tâtonnement adjustment processes (Saari 1985, Bala & Majumdar 1992).

A novel approach to the dynamics of large-scale social systems, evolutionary game theory, was initiated by Maynard Smith & Price (1973), and adapted to dynamical systems theory in subsequent years (Taylor & Jonker 1978, Friedman 1991, Weibull 1995). The application of these models to economics involved the shift from biological reproduction to behavioral imitation as the mechanism for the replication of successful agents.

We apply this framework by treating the Walrasian economy as the stage game of an evolutionary process. We assume each agent is endowed in each period with a good that he must trade to obtain the various goods he consumes. There are no inter-period exchanges. An agent's trade strategy consists of a set of *private prices* for the good he produces and the goods he consumes, such that, according to the individual's private prices, a trade is acceptable if the value of goods received is at least as great as the value of the goods offered in exchange. The exchange process of the economy is hence defined as a multipopulaton game with private prices as strategies. We assume that the strategies of traders are updating according to the replicator dynamic. With rather mild assumptions, the stability of equilibrium is then guaranteed.

2 The Walrasian Economy

We consider an economy with a finite number of goods, $h = 1, \ldots, l$, and a finite number of agents $i = 1, \ldots, m$. Agent i has $\mathbf{R}^l_+$ as consumption space, a utility function $u_i : \mathbf{R}^l_+ \to \mathbf{R}_+$ and an initial endowment $e_i \in \mathbf{R}^l_+$. This economy is denoted by $\mathcal{E}(u, e)$.

In this setting an allocation $\overline{x} \in (\mathbf{R}^l_+)^m$ of goods is *feasible* if it belongs to the set

$$\mathcal{A}(e) = \left\{ x \in (\mathbf{R}^l_+)^m \,\middle|\, \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \right\}.$$

The demand of an agent i is the mapping d_i: $\mathbf{R}^l_+ \to \mathbf{R}^l_+$ that associates to a price $p \in \mathbf{R}^l_+$ the utility-maximizing individual allocations satisfying the budget constraint. That is,

$$d_i(p) := \text{argmax}\{u(x_i) \big| x_i \in \mathbf{R}^l_+, p \cdot x_i \leq p \cdot e_i\}.$$

A feasible allocation $\overline{x} \in \mathcal{A}(e)$ then is an *equilibrium allocation* if there exists a price $\overline{p} \in S$ such that for all i, $\overline{x}_i \in d_i(\overline{p})$. We denote the set of such equilibrium prices by $\mathbf{E}(u, e)$.

To ensure that the economy satisfies conditions for the existence of such an equilibrium allocation, we first define a *quasi-equilibrium*, which consists of an attainable allocation $x^* \in \mathcal{A}(e)$ and a price $p^* \in \mathbf{R}^l_+$ such that $u_i(x_i) > u_i(x_i^*)$ implies $p^* \cdot x_i > p^* \cdot x_i^*$. We say u_i is *locally non-satiated* if for all $x_i \in \mathbf{R}^l_+$, for all $\epsilon > 0$ there exists $x_i' \in \mathbf{R}^l_+ \cap B(x_i, \epsilon)$ such that $u_i(x_i') > u_i(x_i)$. To guarantee the existence of a quasi-equilibrium, it suffices to assume (see Florenzano 2005):

Assumption 1 (Utility) *For all $i = 1, \ldots, m$, u_i is continuous, strictly concave, and locally non-satiated.*

The strict concavity condition in Assumption 1 is not necessary for the existence of a quasi-equilibrium but implies demand mappings are single-valued, which will prove useful below.

To ensure that every quasi-equilibrium is an equilibrium allocation, it suffices to assume that at a quasi-equilibrium the agents do not receive the minimal possible income (see Hammond 1993, Florenzano 2005).

Assumption 2 (Income) *For every quasi-equilibrium (p^*, x^*), and for every $i = 1, \ldots, m$, there exists $x_i \in \mathbf{R}^l_+$ such that $p^* \cdot x_i^* > p^* \cdot x_i$.*

In the following, we assume Assumptions (1) and (2) hold so that the economy has at least an equilibrium. We moreover restrict attention to the generic case (see Balasco 2009) where the economy is regular and has a finite set of equilibria.

3 Exchange Processes with Private Prices

We assume agents determine their behavior on the basis of private prices that represent their subjective priors concerning utility-maximizing exchange ratios for their interactions with other agents in the economy. The distribution of private prices then governs the exchange process and determines the allocation of commodities in the economy. In other words, we consider a game $\mathcal{G}(u, e, \xi)$ such that:

- Each agent has the strategy set $P = K^l$, where $K \subset \mathbf{R}^l_+$ is a finite set of commodity prices with minimum $p_{\min} > 0$ and maximum $p_{\max} > p_{\min}$. Good l is used as a numeraire and its price is fixed equal to 1.
- An exchange mechanism $\xi\colon\ P^m \to \mathcal{A}(e)$ associates to a profile of private prices $\pi = (p_1, \ldots, p_m)$ an attainable allocation $\xi(\pi) = (\xi_1(\pi), \ldots, \xi_m(\pi)) \in \mathcal{A}(e)$ and the payoff of player i then is $\phi_i(\pi) = u_i(\xi_i(\pi))$.

The exchange process ξ is thought to represent the outcome of a sequence of bilateral trades of the kind described in (Gintis 2007, 2012) where agent i only accepts trade which have positive values according to its private price p_i. The price set P is assumed to be constructed in such a way that it contains each of the finite number of equilibrium price profiles of the economy.

In the following, we investigate the relationships between equilibrium allocations of the economy $\mathcal{E}(u, e)$ and Nash equilibria of the game $\mathcal{G}(u, e, \xi)$ when additional restrictions are placed on the exchange mechanism ξ.

4 Strict Equilibria and Dynamic Stability

Our analysis of dynamic stability is based on the replicator dynamic (see Weibull 1995). We recall that given an n-players game with strategy sets $\{S_i | i = 1, \ldots, n\}$ and payoff functions $\{v_i | i = 1, \ldots, n\}$, we have:

Definition 1 *A strategy profile $\{s_i^* | i = 1, \ldots, n\}$ is a* strict Nash equilibrium *if, for all i and for every $s_i \in S_i$ with $s_i \neq s_i^*$, we have $v_i(s_i^*, s_{-i}^*) > v_i(s_i, s_{-i}^*)$.*

We then have (Weibull 1995; see also Appendix A):

Proposition 1 *A strict Nash equilibrium of a multipopulation game is asymptotically stable for all weakly payoff-positive selection dynamics, including the replicator dynamic. Every asymptotically stable equilibrium allocation in the replicator dynamic of a multipopulation game is a strict Nash equilibrium of the game.*

Hence, characterizing exchange mechanisms ξ, such that for every economy $\mathcal{E}(u, e)$ the strict Nash equilibria of the game $\mathcal{G}(u, e, \xi)$ correspond to the equilibrium allocations of the economy $\mathcal{E}(u, e)$ is equivalent to characterize exchange mechanisms for which Walrasian equilibria are the only asymptotically stable states for the replicator dynamic.

5 A Stable Tâtonnement Process with Private Prices

We begin with the simple case of the tâtonement process, which turns out to be stable with private prices under the replicator dynamic. In this process there is no trade unless prices are such that all markets clear. The corresponding exchange mechanism ξ satisfies the following assumption:

Assumption 3 (Tâtonnement) *We call ξ a* tâtonnement *exchange process if*

$$\xi_i(\pi) = \begin{cases} d_i(\overline{p}) & \text{if for all } i,\ \pi_i = \overline{p} \in \mathbf{E}(u,e) \\ e_i & \text{otherwise.} \end{cases} \tag{1}$$

We say that at equilibrium allocation there are *gains from trade* if:

Assumption 4 (Gains from trade) *For every $\overline{p} \in \mathbf{E}(u,e)$, we have for all i, $u_i(d_i(\overline{p})) > u_i(e_i))$.*

Assuming a tâtonnement exchange process in which there are gains from trade at all equilibrium allocations, it is clear that:

- At a strategy profile in which two or more agents use a non-equilibrium price, each agent is allocated his initial endowment and no unilateral deviation can modify this allocation, so that the profile is a non-strict Nash equilibrium

- At a strategy profile in which all agents but one use the same equilibrium price, each agent is allocated his initial endowment. Under Assumptions (4), the "non-equilibrium" agent makes everyone strictly better-of by deviating to the equilibrium price as each agent is then allocated his equilibrium demand. So, such a strategy profile is not a Nash equilibrium.

- At a strategy profile in which all agents use the same equilibrium price $\overline{p}$, each agent receives the equilibrium allocation $u_i(d_i(\overline{p}))$. Under Assumption (4), an agent deviating to a different price makes everyone strictly worse off as each agent is then allocated his initial endowment. Hence, the strategy profile is a strict Nash equilibrium.

It follows that the equilibrium allocations of the economy are the only strict Nash equilibria of the game:

Proposition 2 *Under Assumption (1) and (4), $\pi \in P^m$ is a strict Nash equilibrium of $\mathcal{G}(u,e,\xi)$ if and only if there exists $\overline{p} \in \mathbf{E}(u,e)$ such that for all i, $\pi_i = \overline{p}$.*

Following Proposition 1, this yields the following dynamic stability result for the equilibrium allocation of the underlying economy:

Proposition 3 *Under Assumption (1) and (4), the only asymptotically stable strategy profiles for the replicator dynamic in the game $\mathcal{G}(u,e,\xi)$ are those in which each agent uses an equilibrium price $\overline{p} \in \mathcal{E}(u,e)$ and is allocated the corresponding equilibrium allocation $d_i(\overline{p})$.*

This stable version of the tâtonnement process with private prices is an interesting curiosity, but in a decentralized setting it is implausible that no trade will take place out-of-equilibrium, including ones that are profitable for all the agents involved. The more fundamental issue for us is the evolutionary stability of Walrasian equilibrium in a setting where agents produce and trade out of equilibrium.

6 Stability with Trading Out of Equilibrium

We now suppose that for each agent i there is one good g_i such that the initial endowment of i consists only of good g_i.

Assumption 5 (Production) *For $i = 1, \ldots, m$, there exists $g_i \in \{1, \ldots, l\}$ such that $e_{i,h} > 0 \Leftrightarrow h = g_i$.*

We refer to g_i as agent i's *production good* and call the corresponding price p_{i,g_i} the *production price* of agent i. We assume that agents either do not derive utility from the consumption of their production good, or that they bring to market only the excess over their desired consumption of their production good. With respect to the other goods we shall assume that an agent consumes a fixed subset of goods, which all are necessary for strictly positive consumption utility. This assumption is more general than the standard one in the literature which requires that all goods are consumed in strictly positive quantities. For a subset $C \subset \{1, \cdots, l\}$, we define the vector subspace V_C as $V_C := \{x \in \mathbf{R}^l \mid \forall h \notin C \; x_h = 0\}$ and pr_{V_C} as the projection on V_C.

Assumption 6 (Desirability) *For every $i = 1, \ldots, m$, there exists $C_i \subset \{1, \cdots, l\}$ and $v_i : V_{C_i} \to \mathbf{R}_+$ such that:*

1. *for all $x \in \mathbf{R}^l_+$, $u_i(x) = v_i(pr_{V_{C_i}}(x))$*
2. *$v_i(y) > 0 \Rightarrow \forall h \in C, \; y_h > 0$.*

The standard assumption in the general equilibrium literature is to consider that the excess demand satisfies a boundary condition according to which demand for any good tends to infinity as its price approaches zero (Balasco 2009). As we have a finite price space, we adopt a milder condition stating that when the price of a good is minimal there necessarily is excess demand for that good:

Assumption 7 (Aggregate Nonsatiation) *For all $h = 1, \ldots, l$ and for all $p \in P$, we have: $p_h = p_{\min} \Rightarrow \sum_{i=1}^m d_{i,h}(p) > \sum_{i=1}^m e_{i,h}$.*

We shall from this point onward assume that the Production Assumption (5), the Desirability Assumption (6), and the Aggregate Nonsatiation Assumption (7) are satisfied. Note that in this setting the relevant prices for agent i are limited to that of his production good g_i and those of the set of goods he consumes C_i. Hence, the strategy set of agent i in fact is $P_i := K^{C_I \cup \{g_i\}}$, the product of strategy sets is $\Pi := \prod_{i=1}^{m} P_i$ and if $p \in P$ and $p_i \in P_i$ we write with a slight abuse of notation that $p = p_i$ if and only if $\text{pr}_{P_i}(p) = p_i$.

We look for plausible restrictions on the exchange mechanism ξ that guarantee that the only strict Nash equilibria of the game $\mathcal{G}(u, e, \xi)$ are equilibrium allocations of the economy $\mathcal{E}(u, e)$. This investigation is meaningful only if there is indeed a counterpart to Walrasian equilibrium in the exchange process. The minimal assumption in this respect is that when all agents use the same equilibrium price as private price, the exchange process yields the corresponding equilibrium allocation. That is:

Assumption 8 (Equilibrium) *If $\pi \in \Pi$ is such that for every $i = 1, \ldots, m$, $\pi_i = \overline{p} \in \mathbf{E}(u, e)$, we have for every $i = 1, \ldots, m$, $\xi_i(\overline{p}) = d_i(\overline{p})$.*

The essential role of private prices in the exchange mechanism is to indicate which trades are worth undertaking. We assume an agent purchases a good only if it is sold at a price not greater than his private price:

Assumption 9 (Compatibility) *For all $i = 1, \ldots, m$ and for all $h \neq g_i$, we have $\xi_{i,h}(p) > 0$ only if there exists j such that $g_j = h$ and $p_{i,g_j} \geq p_{j,g_j}$.*

We shall then call a price profile $\pi = (p_1, \ldots, p_m)$ *compatible* if it is such that for all $i = 1, \ldots, m$ and for all $h \in C_i$ there exists j such that $g_j = h$ and $p_{i,g_j} \geq p_{j,g_j}$. We shall denote by $\mathcal{C} \subset \Pi$ the set of compatible price profiles. We will also refer to *h-uniform price profiles* as those where each seller of good h uses the same price. That is $\pi = (p_1, \ldots, p_m)$ is h-uniform if there exists $q \in [p_{\min}, p_{\max}]$ such that for all $i \in \{1, \ldots, m\}$with $g_i = h$, one has $p_{i,g_i} = q$.

We now turn to the price sensitivity of the exchange mechanism. We adopt a mild monotonicity condition that ensures that a uniform decrease of consumption prices increases the utility of an agent:

Assumption 10 (Demand Monotonicity) *Let $\pi = (p_i, p_{-i})$ and $\pi' = (p'_i, p_{-i})$ be two distinct compatible price profiles and $i \in \{1, \ldots, m\}$ such that $p'_{i,g_i} = p_{i,g_i}$ and for $h \in C_i$, either $p'_{i,h} = p_{i,h}$ or π and π' are h-uniform and $p'_{i,h} < p_{i,h}$. Then $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$, the inequality being strict if π' is a price profile for which the Equilibrium Assumption (8) holds.*

Second we assume that each seller is in a symmetric position with regards to market-power, so that if two distinct production prices coexist for the same good, it is profitable for one of the seller to deviate to the price of the other. We write $\pi|p_{i,h} \to p'_{i,h}$ to represent the price profile in which $p_{i,h}$ in π is replaced by $p'_{i,h}$.

Assumption 11 (Seller Symmetry) *Let π be a compatible price profile such that for an $h \in \{1, \cdots l\}$, there exists $i, j \in \{1, \cdots, m\}$ with $g_i = g_j = h$ and $\pi_{i,h} \neq \pi_{j,h}$. Let $\pi' = \pi|\pi_{i,h} \to \pi_{j,h}$, and let $\pi'' = \pi|\pi_{j,h} \to \pi_{i,h}$. Then either $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$ or $u_i(\xi_i(\pi'')) \geq u_i(\xi_i(\pi))$*

Remark 1 *A case where assumption Seller Symmetry is trivially verified is where all potential buyers address their demands to the lowest price seller. Indeed, it is then always advantageous to the agent with the highest selling price, who has income and hence utility zero, to shift to this of the lowest price agent.*

Finally, we shall assume there exists some form of competition among sellers so that in case of excess supply, an agent can increase its market share and be better-off by decreasing its price.

Assumption 12 (Competition) *Let $\pi = (p_i, p_{-i})$ be a compatible price profile such that for an $h \in \{1, \cdots l\}$, $\sum_{j=1}^{n} e_{j,h} > \sum_{j=1}^{m} d_{j,h}(\pi_j)$. There then exists $i \in \{1, \cdots, m\}$ with $g_i = h$ and $p'_i \in P_i$ such that $\pi' = (p'_i, p_{-i})$ is a compatible price profile and $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$.*

Remark 2 *For certain exchange processes ξ, the Competition Assumption (12) can involve the primitives of the economy and the price space. For example, we say that agents j is a* replicate *of agent i f $g_j = g_i$ and $u_i = u_j$. We can ensure that the Competition Assumption (12) is satisfied if we assume that every agent has a replicate and in case of excess supply, demand is allocated first to the seller with the lowest offer price and shared equally among sellers with equal price. Then, provided the price grid is sufficiently fine, in case of excess supply either an agent or its replicate can slightly decrease his production price without being worse off.*

Assumptions (8) through (12) imply that the only strict Nash equilibria of $\mathcal{G}(u, e, \xi)$ are the equilibrium allocations of $\mathcal{E}(u, e)$. Namely:

Proposition 4 *Under Assumptions (8) – (12), $\pi \in \Pi$ is a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if and only if there exists $\overline{p} \in \mathbf{E}(u, e)$ such that for all $i = 1, \ldots, m$, $\pi_i = \overline{p}$.*

Proof: Suppose π is such that for all $i = 1, \ldots, m$, we have $\pi_i = \overline{p}$, where $\overline{p} \in \mathbf{E}(u, e)$. According to Equilibrium Assumption (8), the payoff to each player i, given the strategy profile π, is $u_i(d_i(\overline{p})) > 0$.

Assume agent i deviates to private price p'. If $p'_{g_i} > \overline{p}_{g_i}$ or there exists $h \in C_i$ such that $p'_h < \overline{p}_h$, given that all the other agents still use price $\overline{p}$, we have according to Compatibility Assumption (9) that $\xi_{i,h}(p', \pi_{-i}) = 0$ and hence $u_i(\xi_i(p', \pi_{-i})) = 0$, so that agent i is strictly worse off. Otherwise, one has for all $h \in C_i$, $p'_h \geq \overline{p}_h$ with at least one inequality being strict, given that $p' \neq \overline{p}$. Demand Monotonicity Assumption (10) then implies that $u_i(\xi_i(p', \pi_{-i})) < u_i(\xi_i(\overline{p}, \pi_{-i}))$, so that agent i is strictly worse off. This proves that π is a strict Nash equilibrium of the game $\mathcal{G}(u, e, \xi)$.

Conversely, let us show that if $\pi \in \Pi$ is such that for at least an $i \in \{1, \ldots, m\}$ $\pi_i \notin \mathbf{E}(u, e)$ then $\pi \in \Pi$ is not a strict Nash equilibrium.

First consider the case where there exists $\tilde{p} \notin \mathbf{E}(u, e)$ such that for all i, $\pi_i = \tilde{p}$. Using Walras' law, one has $\tilde{p} \cdot \sum_{i=1}^{m}(d_i(\tilde{p}) - e_i) = 0$. As $\tilde{p}$ is not an equilibrium price and $\tilde{p} \in \mathbf{R}^l_{++}$, we must have for some h, $\sum_{j=1}^{h} d_{j,h}(\tilde{p}) < \sum_{j=1}^{h} e_{j,h}$. Consider then an agent $i \in \{1, \cdots, m\}$ such that $g_i = h$. According to Competition Assumption (12), this agent can deviate to a price p' without being worse off. This implies π is not a strict Nash equilibrium of the game $\mathcal{G}(u, e, \xi)$.

Second, consider the case where π is such that there exists $i, j \in \{1, \ldots, m\}$ with $\pi_i \neq \pi_j$ or equivalently that there exists i, j with $\pi_{i,g_j} \neq \pi_{j,g_j}$. We consider three subcases.

- A first case is where the price profile is not compatible. That is for at least one $i \in \{1, \ldots, m\}$ and one $h \in C_i$, one has for all j with $g_j = h$, $\pi_{i,h} < \pi_{j,h}$. The Compatibility Assumption (9) then yields that $\xi_{i,g_j}(\pi) = 0$ and $u_i(\xi_i(\pi)) = 0$, so that π cannot be a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$.

- A second case is where the price profile is compatible and one agent has a private price for a good greater than that of each potential seller, who all use the same price. That is there is $i \in \{1, \ldots, m\}$ and $h \in C_i$ such that for all j, j' with $g_j = g_{j'} = h$, $\pi_{i,h} \geq \pi_{j,h} = \pi_{j',h}$, with one of those inequalities being strict. Let then i and h be such that for all j with $g_j = h$, $\pi_{i,h} > \pi_{j,h}$. Suppose agent i deviates to p' such that $p'_h = \pi_{j,h}$ and for all $h' \neq h$, $p'_{h'} = \pi_{i,h}$. It is a direct consequence of the Demand Monotonicity Assumption (10) that $u_i(\xi_i(\pi)) \leq u_i(\xi_i(p', \pi_{-i}))$ so that π cannot be a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$.

- The final case is where the price profile is compatible and two sellers use a different price. That is $\pi \in \mathcal{C}$ and there exists $h' \in \{1, \ldots, l\}$ and $k, \ell \in \{1, \ldots, m\}$ such that $g_k = g_\ell = h'$ and $\pi_{k,h'} \neq \pi_{\ell,h'}$. In this setting, the Seller Symmetry Assumption (11), guarantees that an agent can deviate without being worse off, so that π cannot be a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$.

As in the case of Proposition (1), Proposition (4) admits a dynamical counterpart:

Proposition 5 *The only asymptotically stable strategy profiles for the replicator dynamic in $\mathcal{G}(u, e, \xi)$ are those for which each agent uses an equilibrium price $\overline{p} \in \mathcal{E}(u, e)$ and agent i is allocated his equilibrium allocation $d_i(\overline{p})$.*

Remark 3 *Assumption Competition (12) can be replaced by an alternative involving excess demand. One could assume that if $\pi = (p_i, p_{-i})$ is a compatible price profile such that for an $h \in \{1, \cdots l\}$, $\sum_{j=1}^{n} e_{j,h} < \sum_{j=1}^{m} d_{j,h}(\pi_j)$, then there exists $i \in \{1, \cdots, m\}$ with $h \in C_i$ and $p'_i \in P_i$ such that $\pi' = (p'_i, p_{-i})$ is a compatible price profile and $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$. An example of sufficient condition on ξ for the latter assumption to hold is that the price grid be sufficiently fine and that in case of excess demand, agents with the highest buying price have their demand satisfied first while rationing is symmetric when all buyers have the same price.*

7 Examples of Stable Exchange Processes

It is straightforward to check that the tâtonnement exchange process defined in Section 5 satisfies the necessary conditions for the application of Proposition 4.

Another example of trading process satisfying the conditions of Proposition 4, assuming there are at least two producers of each good, is the following.

- Agents whose private price is not compatible with the price profile are allocated their initial endowment.
- Agents are ordered in such a way that if two producers of the same good have a distinct production price then the agent with the lowest price comes first. We denote by $\sigma(i)$ the position of the i^{th} agent in this ordering and by $\sigma(k, h)$ that of the k^{th} producer of good h.
- The income of the first producer of good h, $\sigma(1, h)$, is computed as the maximal amount that can be raised by fulfilling demand at the buyers's prices. That is, if τ_j denotes the agent with the j^{th} highest buying price for good h, the income of agent $\sigma(1, h)$ is

$$\sum_{j=1}^{n} \pi_{\tau_j,h} \max\left(\min\left(d_{\tau_j,h}(\pi_{\tau_j}), e_{\sigma(1,h)} - \sum_{r=1}^{j-1} d_{\tau_r,h}(\pi_{\tau_r})\right), 0\right).$$

The income of the second supplier is then computed as the maximum value that can be raised by fulfilling the remaining demand, up to the amount corresponding to the supplier's endowment, at the buyers's prices. The income of the remaining suppliers are defined accordingly.

- Define then feasible allocations for the first agent in the trade ordering, $\sigma(1)$, as $X_1(\pi) := \{x_1 \in \mathbf{R}_+^{C_1} \mid x_1 \leq \sum_{i=1}^n e_i,\ \pi_1 \cdot x_1 \leq w_1(\pi)\}$ and assume that the agent chooses the utility maximizing allocation $\overline{x}_1$ in this set, so that $\xi_{\sigma(1)}(\pi) = \overline{x}_1$. The set of feasible allocations and the allocation to the k^{th} agent in the ordering, $\sigma(k)$, are then defined by recursion as $X_k := \{x_k \in \mathbf{R}_+^{C_k} \mid x_k \leq \sum_{i=1}^n e_i - \sum_{j=1}^{k-1} \overline{x}_j,\ \pi_k \cdot x_k \leq w_k\}$ and $\xi_{\sigma(k)}(\pi) = \overline{x}_k$ is the utility maximizing allocation in X_k.

- It is straightforward to check that this exchange process satisfies the Equilibrium (8) and Compatibility (9) assumptions.

- Concerning the Demand Monotonicity Assumption (10), note that if π, π' and i are as in Assumption (10), then $X_{\sigma^{-1}(i)}(\pi) \subset X_{\sigma^{-1}(i)}(\pi')$ so that one necessarily has $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$.

- Now let π and h be as in the Competition Assumption (12). Let $j = \sigma(1, h)$ and $i = \sigma(2, h)$. Assume agent i shifts his price to p'_i such that $p'_{i,h} < p_{j,h}$, all other prices remaining equal. Than for the price profile $\pi' = (p'_i, \pi_{-i})$, agent i has its income and consumption prices unchanged but comes beforehand in the trade-ordering. Hence $X_{\sigma^{-1}(i)}(\pi) \subset X_{\sigma^{-1}(i)}(\pi')$ so that one necessarily has $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$.

- Finally, let π and h be as in the Seller Symmetry Assumption (11) and let i, j be two sellers of good h with different prices. As in the case of the Competition Assumption (12), it suffices to consider that the agent that comes second in the trading process deviates to a production price lower than this of the first to show that the assumption is satisfied.

8 Conclusion

Our proof can be summarized as follows. The equilibrium of a Walrasian market system is a strict Nash equilibrium of an exchange game in which the requirements of the exchange process are quite mild and easily satisfied. Assuming producers update their private price profiles periodically by adopting the strategies of more successful peers leads to a multipopulation game in which strict Nash equilibria are asymptotically stable in the replicator dynamic. Conversely, all stable equilibria of the replicator dynamic are strict Nash equilibria of the exchange process.

The major innovation of our model is the use of private prices, one set for each agent, in place of the standard assumption of a uniform public price faced by all agents. The traditional public price assumption would not have been auspicious even had a plausible stability theorem been available using such prices. This is because there is no mechanism for prices to change in a system of public prices—no agent can alter the price schedules faced by the large number of agents with whom any one agent has virtually no contact.

The private price assumption is the only plausible assumption for a fully decentralized market system not in equilibrium, because there is in fact no natural way to define a common price system except in equilibrium. With private prices, each individual is free to alter his price profile at will, market conditions alone ensuring that something approximating a uniform system of prices will prevail in the long run.

There are many general equilibrium models with private prices in the literature, based for the most part on strategic market games (Shapley & Shubik 1977, Sahi & Yao 1989, Giraud 2003) in which equilibrium prices are set on a market-by-market basis to equate supply and demand, and it is shown that under appropriate conditions the Nash equilibria of the model are Walrasian equilibria. These are equilibrium models, however, without known dynamical properties, and unlike our approach they depend on an extra-market mechanism to balance demand and supply.

The equations of our dynamical system are too many and too complex to solve analytically or to estimate numerically. However, it is possible to construct a discrete version of the system as a finite Markov process. The link between stochastic Markov process models and deterministic replicator dynamics is well documented in the literature. Helbing (1996) shows, in a fairly general setting, that mean-field approximations of stochastic population processes based on imitation and mutation lead to the replicator dynamic. Moreover, Benaim & Weibull (2003) show that large population Markov process implementations of the stage game have approximately the same behavior as the deterministic dynamical system implementations based on the replicator dynamic. This allows us to study the behavior of the dynamical market economy for particular parameter values. For sufficiently large population size, the discrete Markov process captures the dynamics of the Walrasian economy extremely well with near certainty (Benaim & Weibull 2003). While analytical solutions for the discrete system exist (Kemeny & Snell 1969, Gintis 2009), they also cannot be practically implemented. However, the dynamics of the Markov process model can be studied for various parameter values by computer simulation (Gintis 2007, 2012).

Macroeconomic models have been especially handicapped by the lack of a general stability model for competitive exchange. The proof of stability of course does not shed light on the fragility of equilibrium in the sense of its susceptibility to exogenous shocks and its reactions to endogenous stochasticity. These issues can be studied directly through Markov process simulations, and may allow future macroeconomists to develop analytical microfoundations for the control of excessive market volatility.

Appendix A: Asymptotic stability and replicator dynamics

Let G be an n-player game with finite strategy sets $\{S_i|i = 1, \ldots, n\}$, the cardinal of which is denoted by $k_i = |S_i|$, with strategies indexed by $h = 1, \ldots, k_i$ and payoff functions $\{\pi_i|i = 1, \ldots, n\}$. Let $\Delta_i = \{\sigma_i \in \mathbf{R}^{k_i}|\forall h, \sigma_{i,h} \geq 0$ and $\sum_{h=1}^{k_i} \sigma_{i,h} = 1\}$ the which is the mixed strategy space of agent i, and let $\Delta = \prod_{i=1}^{n} \Delta_i$. In an evolutionary game setting, an element $\sigma_i \in \Delta_i$ represents a population of players i with a share $\sigma_{i,h}$ of the population playing strategy $h \in S_i$.

Dynamics for such population of players $(\sigma_1, \ldots, \sigma_N) \in \Delta$ are defined by specifying, a growth rate function $g : \Delta \to \mathbf{R}^{\sum_{i=1}^{n} k_i}$, for all $i = 1, \ldots, n$ and $h = 1, \ldots, k_i$:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h} g_{i,h}(\sigma) \tag{2}$$

We shall restrict attention to growth-rate functions that satisfy a regularity condition and maps Δ into itself (Weibull 1995).

Definition 2 *A regular growth-rate function is a Lipschitz continuous function g defined in a neighborhood of Δ such that for all $\sigma \in \Delta$ and all $i = 1, \ldots, n$ we have has $g_i(\sigma) \cdot \sigma_i \neq 0$.*

The dynamics of interest in a game-theoretic setting are those that satisfy minimal properties of monotonicity with respect to payoffs. Strategies of player i in $B_i(\sigma) := \{s \in S_i|u_i(s, \sigma_{-i}) > u_i(\sigma)\}$ that have above average payoffs against σ_{-i}, have a positive growth-rate in the following sense:

Definition 3 *A regular growth-rate function g is weakly payoff-positive if for all $\sigma \in \Delta$ and $i = 1, \ldots, n$,*

$$B_i(\sigma) \neq 0 \Rightarrow g_{i,h} > 0 \textit{ for some } s_{i,h} \in B_i(\sigma), \tag{3}$$

where $s_{i,h}$ denotes the h^{th} pure strategy of player i.

Among the class of weakly-payoff positive dynamics, the replicator dynamic is by far the most commonly used to represent the interplay between population dynamics and strategic interactions. It corresponds to the system of differential defined for all $i = 1, \ldots, n$ and $h = 1, \ldots, |S_i|$ by:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h}(\pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)). \tag{4}$$

That is thus the system of differential equation corresponding to the growth rate function $g_{i,h}(\sigma) = \pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)$.

It is standard to show that the system of differential equations (2) associated with a regular and weakly-payoff monotonic growth function has a unique solution defined at all times for every initial condition in Δ. We will generically denote the solution mapping by $\psi : \mathbf{R}_+ \times \Delta \to \Delta$, so $\psi(t, \sigma_0)$ gives the value at time t of the solution to (2) with initial condition $\sigma(0) = \sigma_0$. Stability properties of (2), are then defined in terms of this solution mapping:

Definition 4 *A strategy profile $\sigma^* \in \Delta$ is called Lyapunov stable if every neighborhood V of σ^* contains a neighborhood W of σ^* such that $\psi(t, \sigma) \in V$ for all $\sigma \in W \cap \Delta$.*

Definition 5 *A strategy profile $\sigma^* \in \Delta$ is called asymptotically stable if it is Lyapunov stable and there exists a neighborhood V of σ^* such that for all $\sigma \in V \cap \Delta$:*

$$\lim_{t \to +\infty} \psi(t, \sigma) = \sigma^*.$$

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